Intrinsic momentum anisotropy for relativistic particles from quantum mechanics

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continuation of DM, Wang, Greene, arXiv:1404.4119

Outline

I. General idea

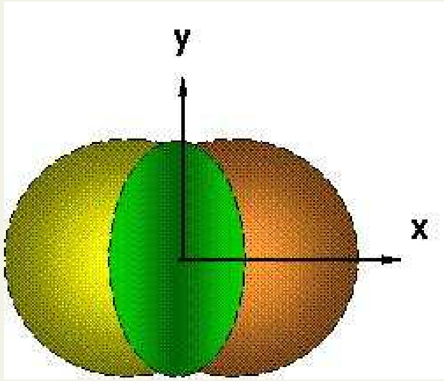
II. Nonrelativistic calculation for 2D harmonic oscillator

III. Recalculation for massless particles

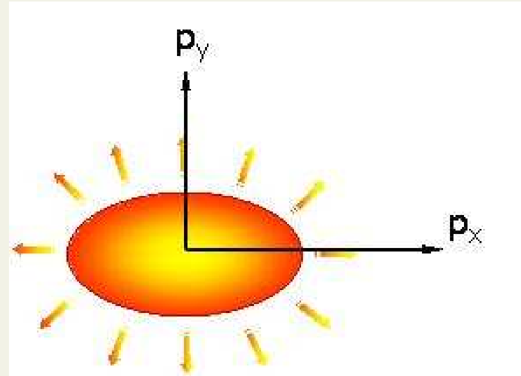
IV. Summary and open questions

Elliptic flow

initial spatial anisotropy converts to final momentum space anisotropy



$\rightarrow$



$$\varepsilon \equiv \frac{\langle x^2 - y^2 \rangle}{\langle x^2 + y^2 \rangle} \qquad\qquad v_2 \equiv \frac{\langle p_x^2 - p_y^2 \rangle}{\langle p_x^2 + p_y^2 \rangle} \equiv \langle \cos 2\phi_p \rangle$$

common picture: hydro - v_2 generated by pressure gradents

$\rightarrow$ **one then uses data to extract matter properties**

BUT other sources: - anisotropic escape in transport e.g., He et al, PLB 753, 506 (2016)
- QCD matrix elements e.g., Dumitru et al, PLB 743, 134 (2015)
- anisotropy from quantum mechanics DM et al, arXiv:1404.4119
- hadron-string dynamics Casing, Bratkovskaya et al **...**

Momentum anisotropy from quantum mechanics

Back of the envelope:

$$v_2 \sim \frac{\langle p_x^2 - p_y^2 \rangle}{\langle p_x^2 + p_y^2 \rangle}$$

from uncertainty relation (for ground state, with $\hbar = 1$):

$$\langle p_x^2 \rangle \;\sim\; 1/R_x^2 \;, \quad \langle p_x^2 \rangle \;\sim\; 1/R_y^2$$

$$\Rightarrow v_2 \sim \frac{R_y^2 - R_x^2}{R_y^2 + R_x^2} = \varepsilon \quad (!)$$

Statistical physics (ideal gas in trap)

$$H = \sum_i H_1(\mathbf{p}_i, \mathbf{r}_i) \ , \qquad H_1(\mathbf{p}, \mathbf{r}) = K(\mathbf{p}) + V(\mathbf{r})$$

Classically, smooth integrals:

$$\frac{dN}{d\mathbf{p}} = N \frac{\int d\mathbf{r}\, e^{-H_1(\mathbf{p},\mathbf{r})/T}}{\int d\mathbf{r}\, d\mathbf{p}\, e^{-H_1(\mathbf{p},\mathbf{r})/T}} = N \frac{e^{-K(\mathbf{p})/T}}{\int d\mathbf{p}\, e^{-K(\mathbf{p})/T}} = isotropic \qquad \Rightarrow \ v_n \equiv 0$$

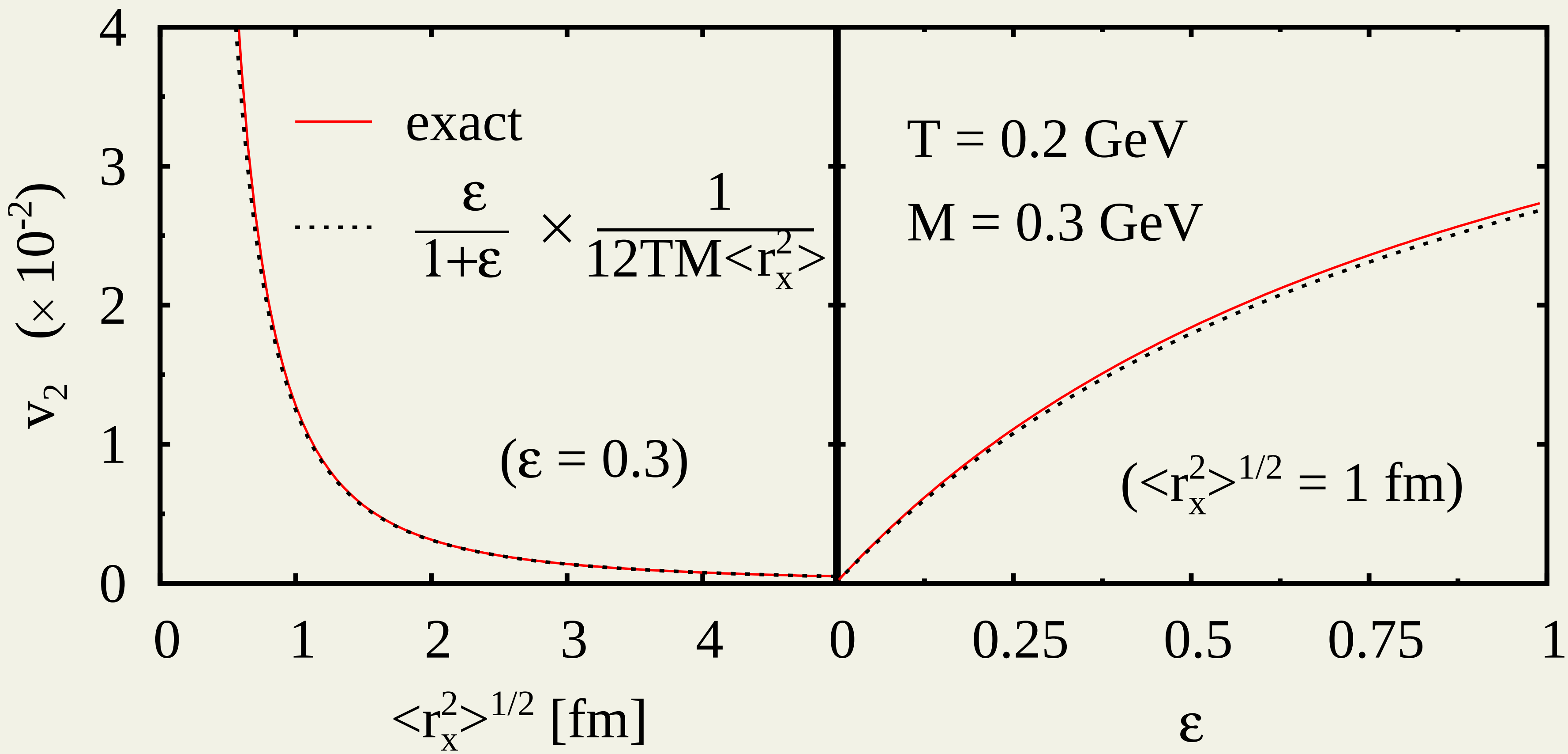
But in QM, level spacing matters:

$$f(\mathbf{p}) \equiv \frac{dN}{d\mathbf{p}} = \frac{1}{Z} \sum_j |\psi_j(\mathbf{p})|^2 \, e^{-E_j/T} = anisotropic$$

for 2D harmonic oscillator $V = \frac{M}{2} \sum \omega_i^2 r_i^2$**,** $f(\mathbf{p})$ **is Gaussian, and** arXiv:1404.4119

$$v_2 \approx \frac{\hbar^2}{12 k_B T M \langle r_x^2 \rangle} \cdot \frac{\varepsilon}{1+\varepsilon} = \frac{\hbar^2}{12 p_{th}^2 \langle r_x^2 \rangle} \cdot \frac{\varepsilon}{1+\varepsilon}$$

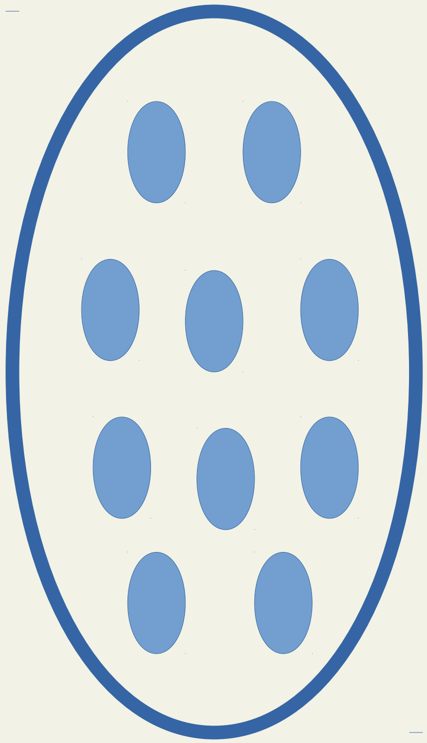
Nonzero. Vanishes in the $T \to 0$ **or** $M \to \infty$ **or** $size \to \infty$ **limits.**

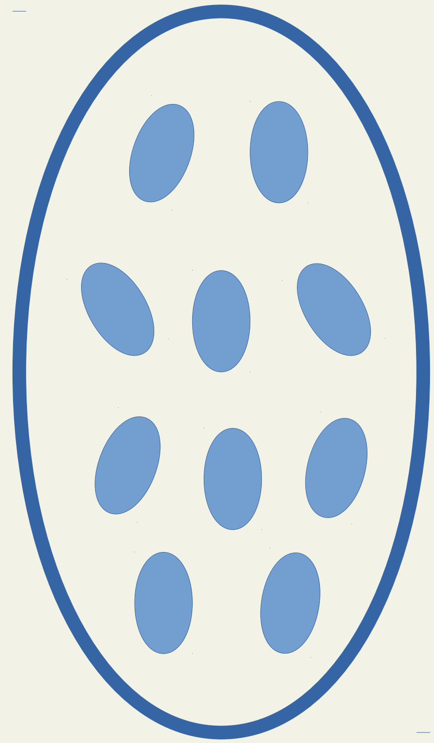


expect percent-level v_2 in small systems (p+A)

also, for not too low T: $v_2(p_T) \propto p_T^2/MT$ **and** $v_{2n}(p_T) \approx [v_2(p_T)]^n/n!$

for hot spots, it can get even better (be it initconds, or freezeout HS)





area $\sim \times 1/N$

size $\sim \times 1/\sqrt{N}$

$v_2 \sim 1/L^2 \sim \times N$

survives $1/\sqrt{N}$ **weakening if orientations fluctuate**

$v_2 \sim \times\sqrt{N}$

This is just a simple wave effect, so classical Yang-Mills should show it too.

Interestingly, $v_2 \neq 0$, but there is no hydrodynamic flow anywhere:

$$\mathcal{L} = \frac{i\hbar}{2}\left(\psi^*\dot{\psi} - \dot{\psi}^*\psi\right) - \frac{\hbar^2}{2M}(\nabla\psi^*)(\nabla\psi) - V(\mathbf{r},t)\psi^*\psi$$

apply Noether's theorem:

$$T^{00} = \frac{\hbar^2}{2M}(\nabla\psi^*)(\nabla\psi) + V(\mathbf{r},t)\psi^*\psi \quad (1)$$

$$T^{0i} = \frac{i\hbar}{2}\left(\psi\nabla_i\psi^* - \psi^*\nabla_i\psi\right) \quad (2)$$

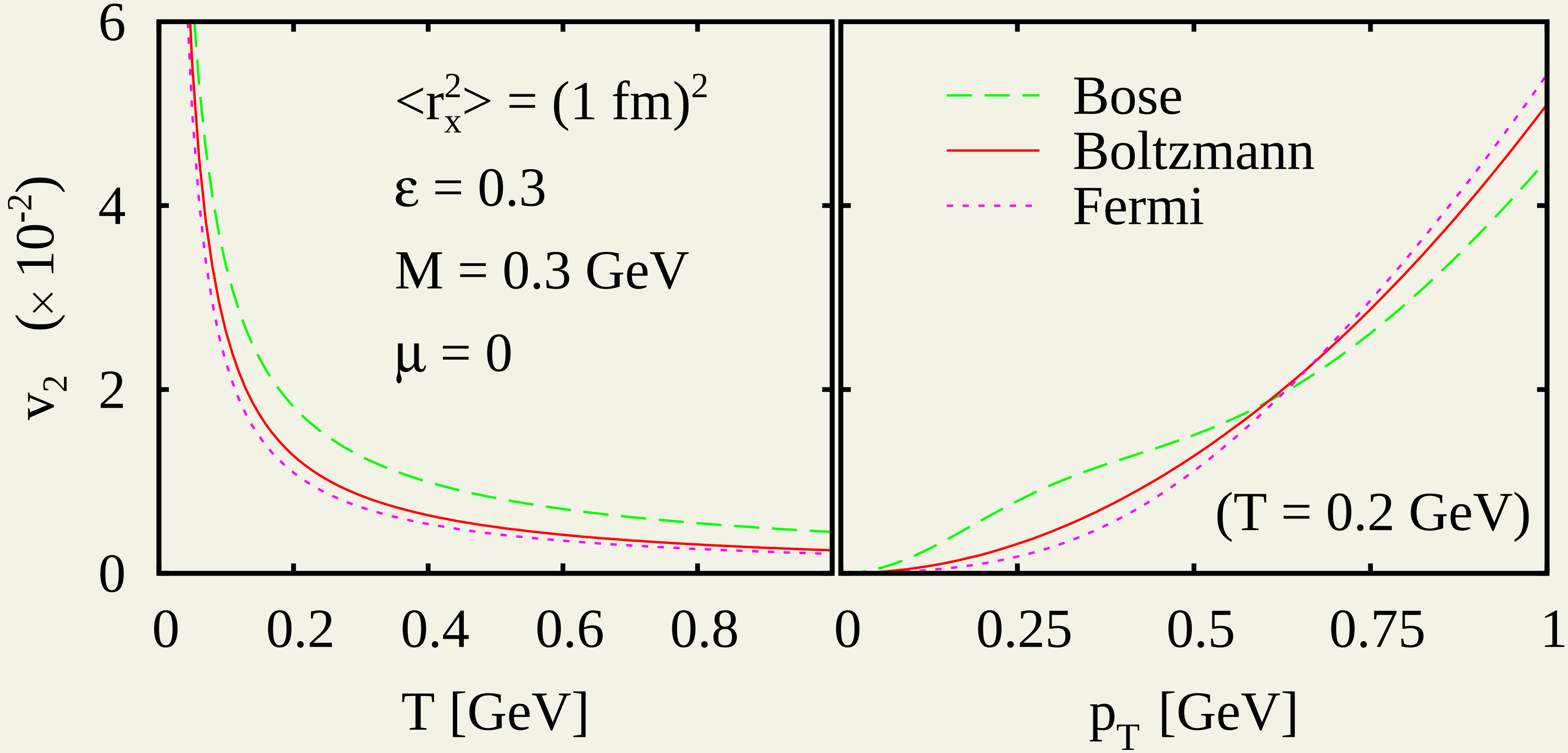
$$T^{i0} = \frac{i\hbar}{2M}\left(\frac{\hbar^2}{2M}\Delta\psi - V\psi\right)(\nabla_i\psi^*) + c.c. \quad (3)$$

$$T^{ij} = \frac{\hbar^2}{2M}\left\{(\nabla_i\psi^*)(\nabla_j\psi) - \frac{1}{2}\delta_{ij}\left[\psi^*\Delta\psi + (\nabla\psi^*)(\nabla\psi)\right]\right\} + c.c. \quad (4)$$

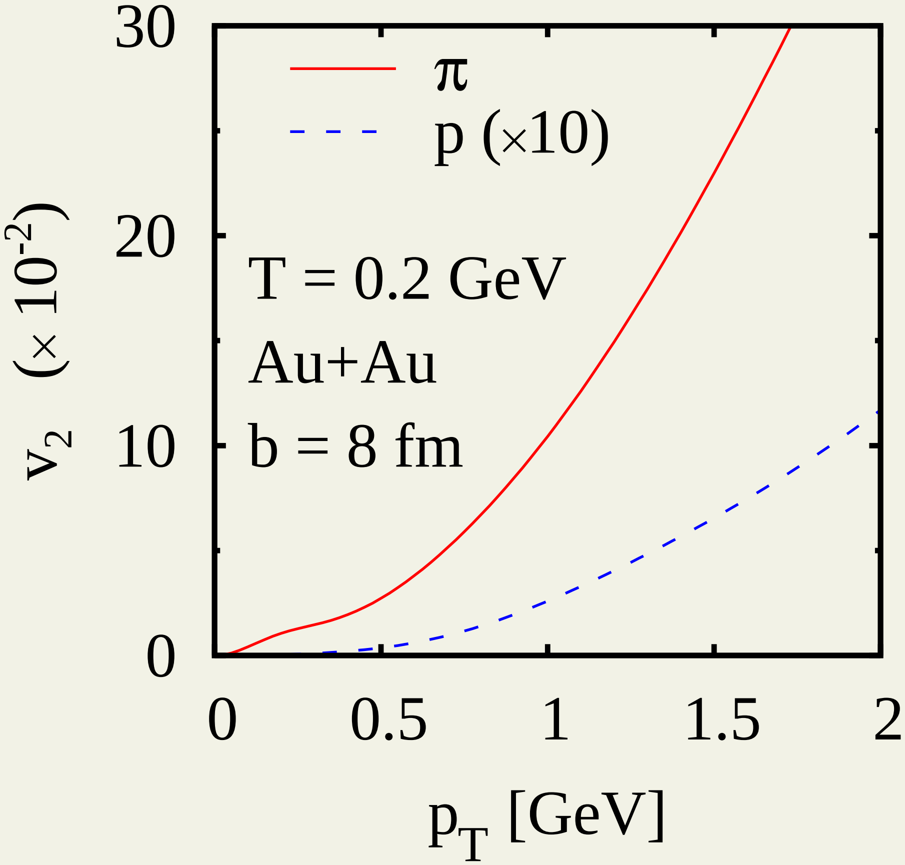
The HO wave functions are real $\Rightarrow T^{0i} \equiv 0 \equiv T^{i0}$

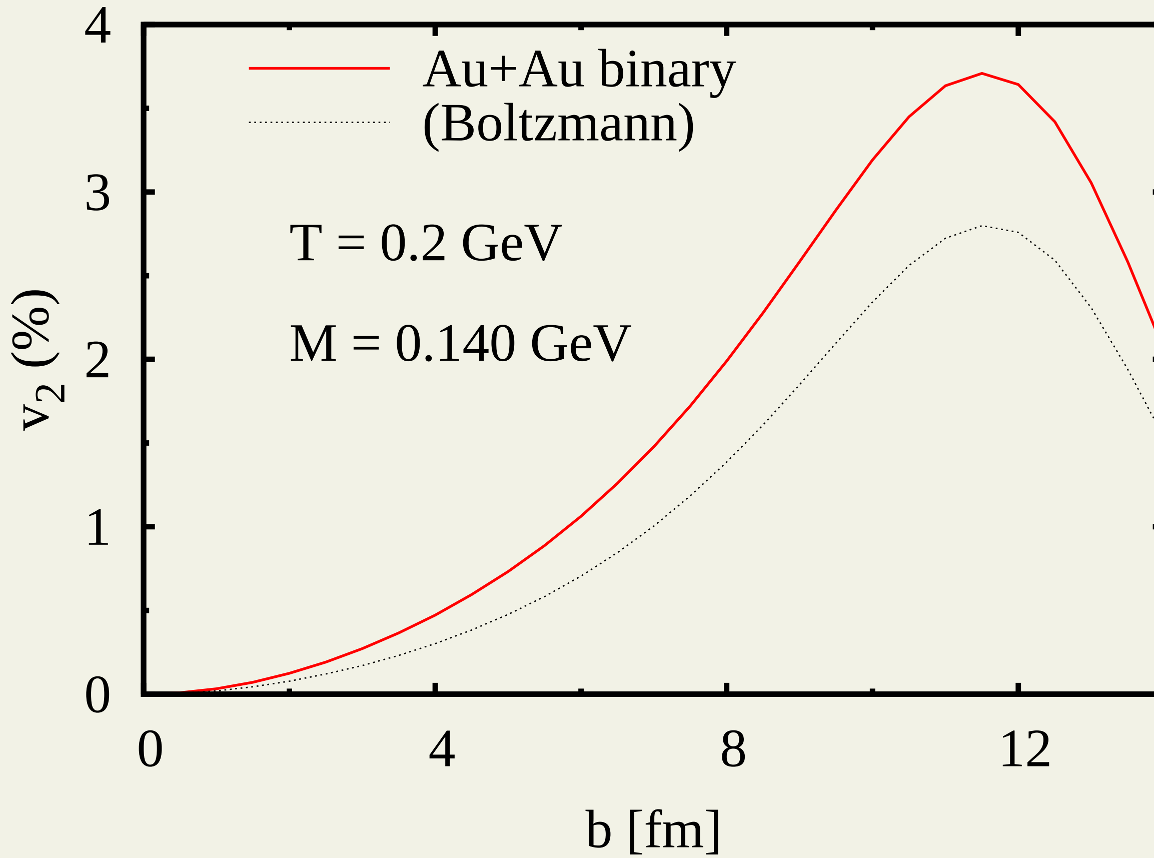
Estimate for HICs

can do Fermi/Bose statistics: $e^{-E_j/T} \to \dfrac{\gamma}{e^{(E_j-\mu)/T}+a}$ $\quad (a = \pm 1)$



Au+Au: significant intrinsic v_2 for light hadrons   1404.4119





with notable caveats: nonrelativistic treatment, no expansion dynamics

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Repeat for relativitic case

try ultrarelativistic limit (massless $m \to 0$):

$$H \equiv K + V = \sqrt{p_x^2 + p_y^2 + m^2} + \mu^3[(1+\alpha)r_x^2 + (1-\alpha)r_y^2]$$

trick: swap p and x, and rescale

$$\bar{r}_{x,y} \equiv -\frac{p_{x,y}}{\mu\sqrt{1 \pm \alpha}} \ , \qquad \bar{p}_{x,y} \equiv \mu\sqrt{1 \pm \alpha}\, r_{x,y}$$

preserves commutation relations $[\bar{r}_i, \bar{p}_j] = i\delta_{ij}$, and Hamiltonian becomes

$$H = \mu\left[\bar{p}_x^2 + \bar{p}_y^2 + \sqrt{(1+\alpha)\bar{r}_x^2 + (1-\alpha)\bar{r}_y^2}\right] \equiv \mu[\bar{K} + \bar{V}]$$

$\to$ one has to find eigenvalues and eigenvectors numerically

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Diagonalize in finite basis

expand over finite basis $|\psi_j\rangle = \sum_n c_{j,n}|\phi_n\rangle$

Schrödinger equation then becomes generalized eigenvalue problem

$$\sum_n \bar{H}_{mn} c_{j,n} = \bar{E}_j \sum_n O_{mn} c_{j,m} \ , \qquad (\bar{H}_{mn} \equiv \langle\phi_m|\bar{H}|\phi_n\rangle \ , \ O_{nm} \equiv \langle\phi_m|\phi_n\rangle)$$

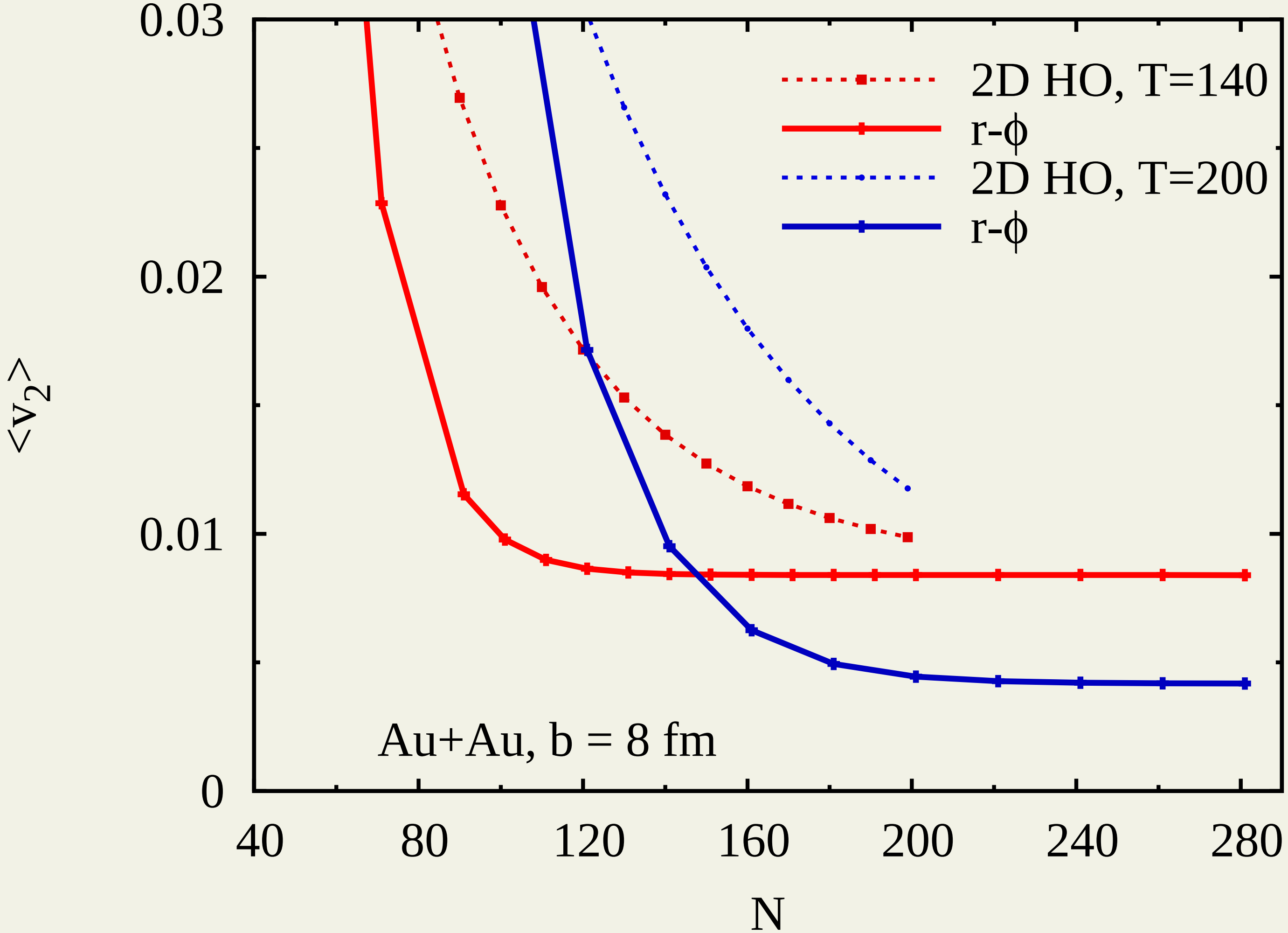
we tried:

- **2D H.O. wave fns** $\psi_n(x)\psi_m(y)$**: orthonormal but expensive to evaluate**
- **factorized** $R_n(r)\phi_m(\phi)$ **in polar coordinates with** $\phi_m(\phi) = e^{im\phi}$

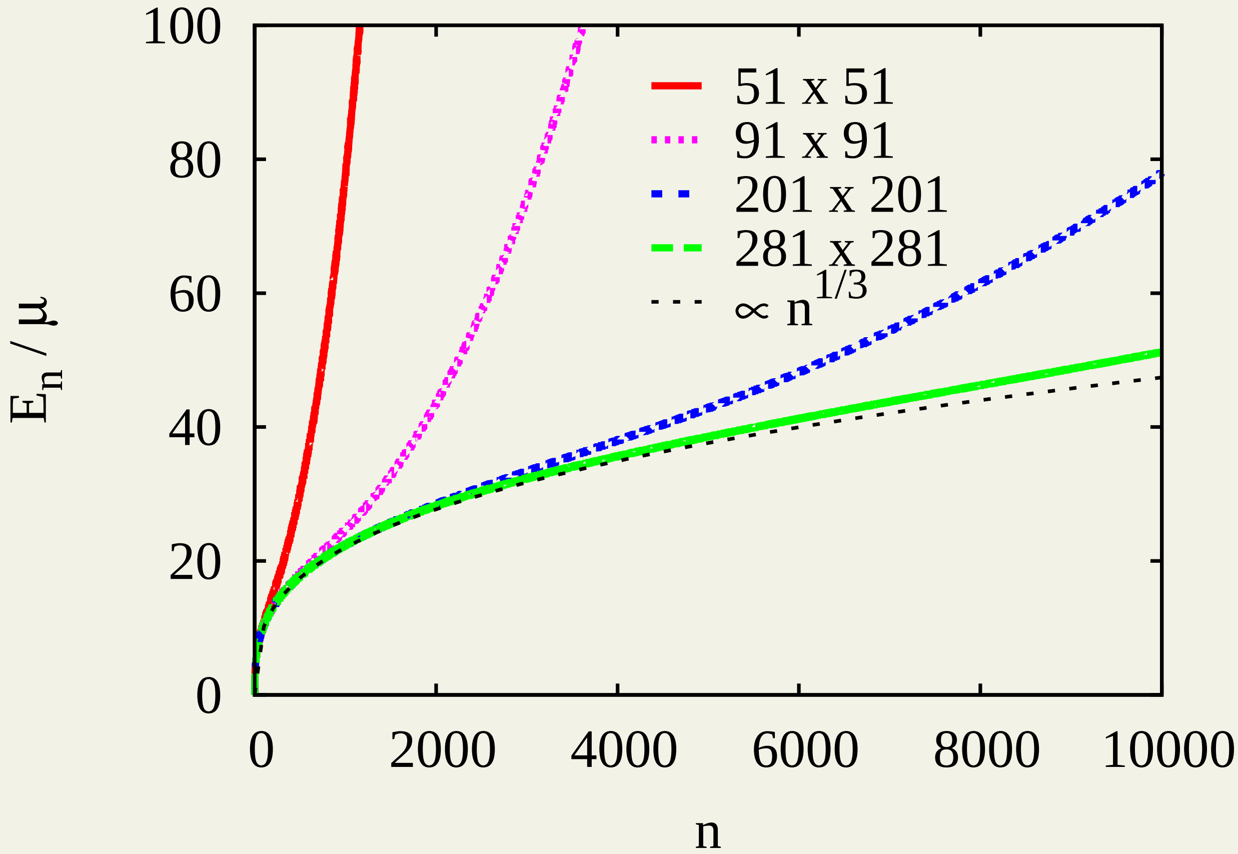
  and $R_n(r) \propto r^n e^{-r}$**: poor for diagonalization, many** $O_{mn} \sim \mathcal{O}(1)$

  and $R_n(r) \sim$ **B-spline basis: good because** O_{mn} **elems often zero**

convergence with $N \times N$ basis states

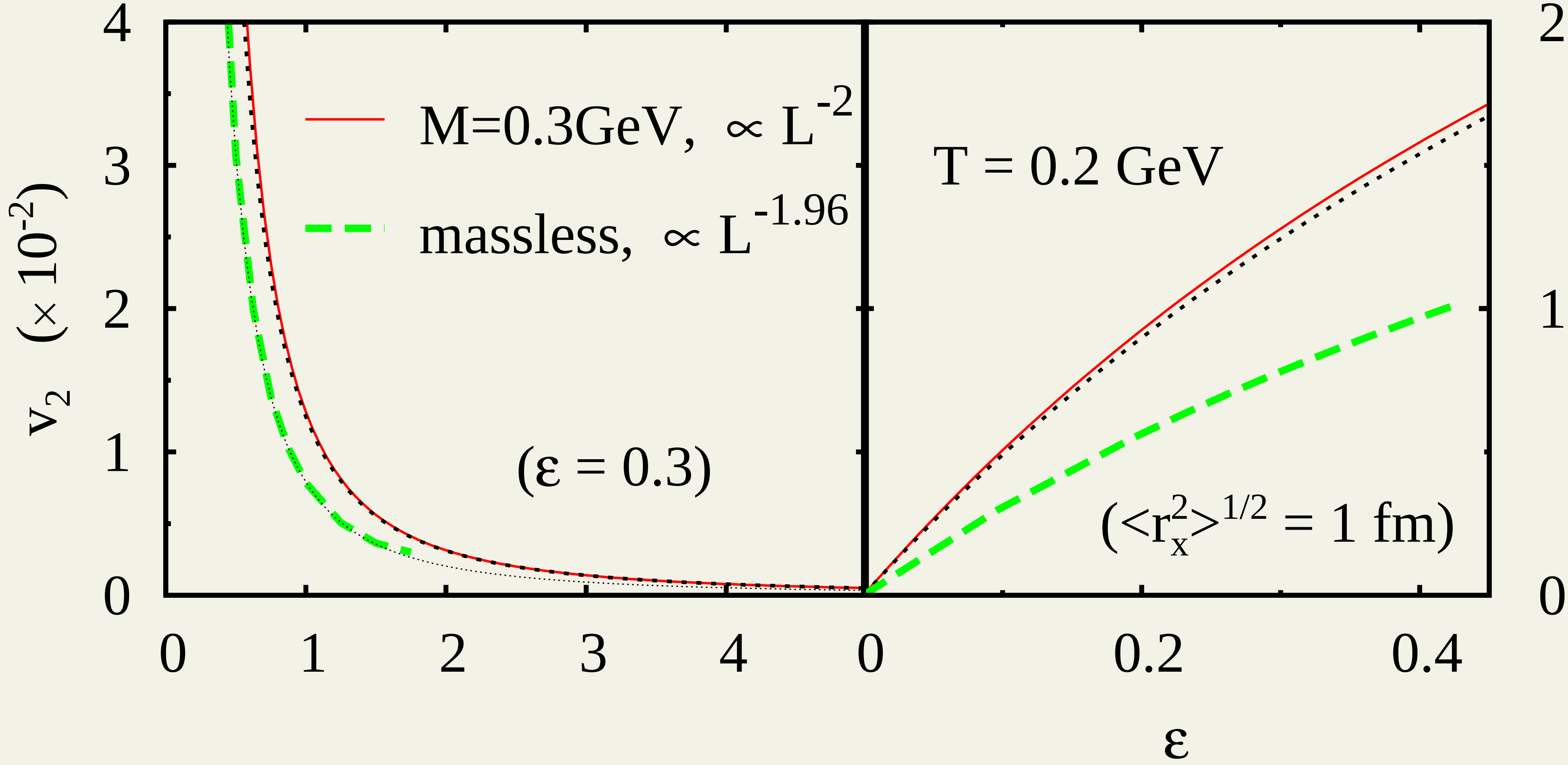


$\langle v_2 \rangle$ overestimated in too small bases



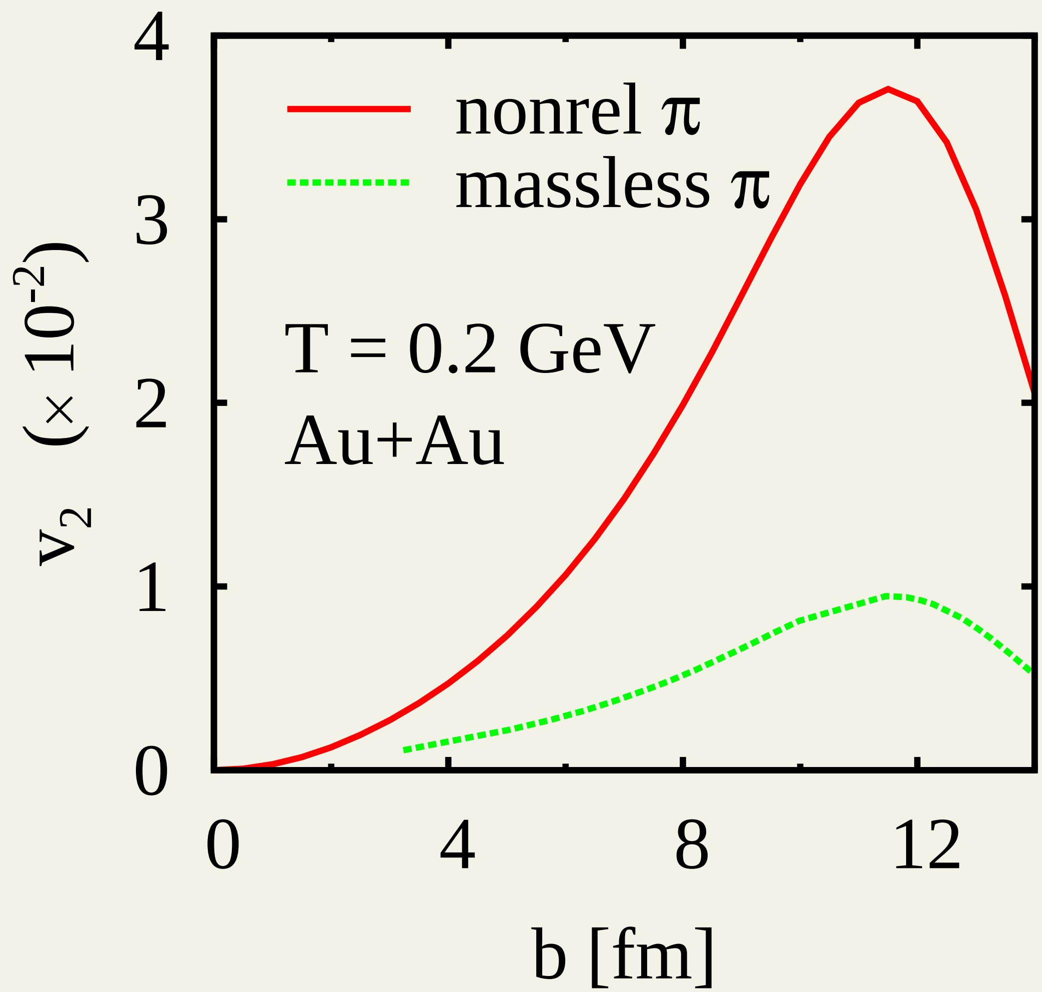
Model	$N_{states}(\leq E)$	dN/dE	ΔE spacing	v_2
2D harmonic oscillator	$\sim E^2$	$\sim E$	$\sim 1/E$	$\sim \frac{1}{L^2 MT}$
Infinite 2D well	$\sim E$	$\sim const$	$\sim const$	$\sim \frac{1}{L\sqrt{MT}}$
Massless 2D	$\sim E^3$	$\sim E^2$	$\sim 1/E^2$	$\sim \frac{1}{L^2}$

averaged v_2 from nonrelativistic vs massless



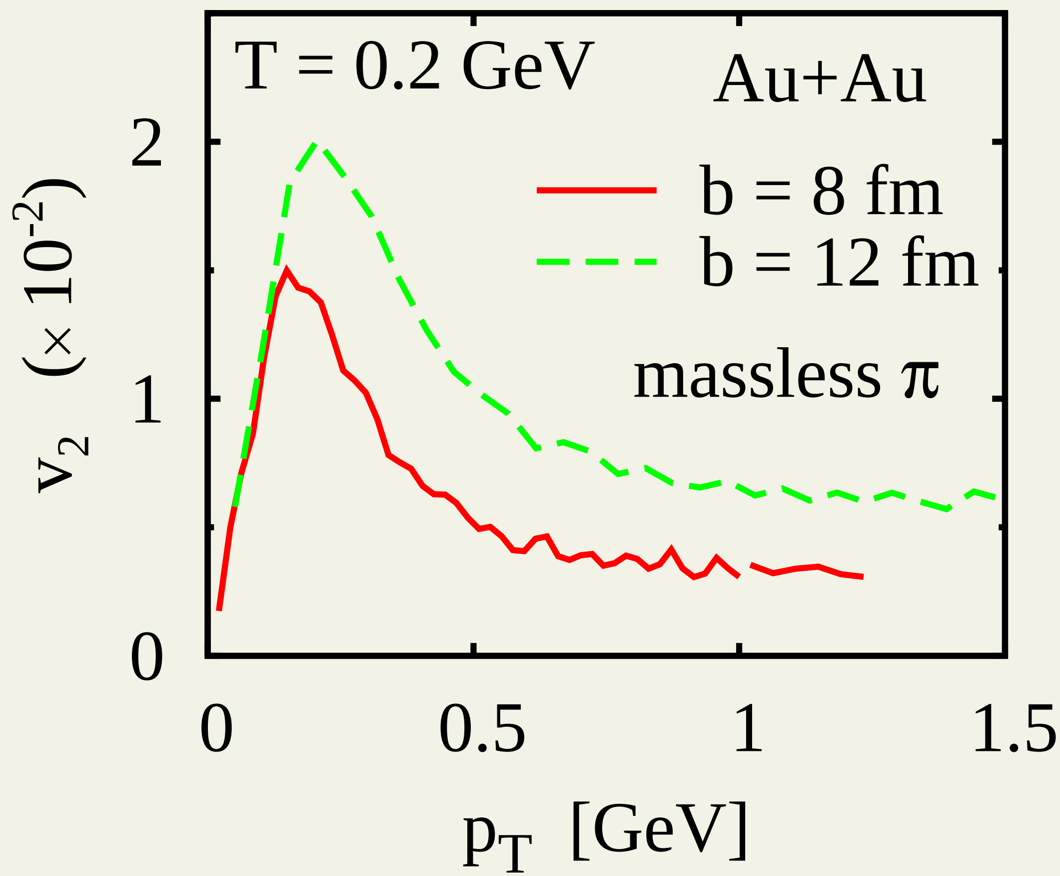
similar $1/L^2$ size dependence for massless, but smaller v_2

For Au+Au: intrinsic v_2 still present



$\sim 3\times$ **smaller in massless limit than from nonrel. approx**

p_T **dependence in massless limit is quite different from NR calculation**

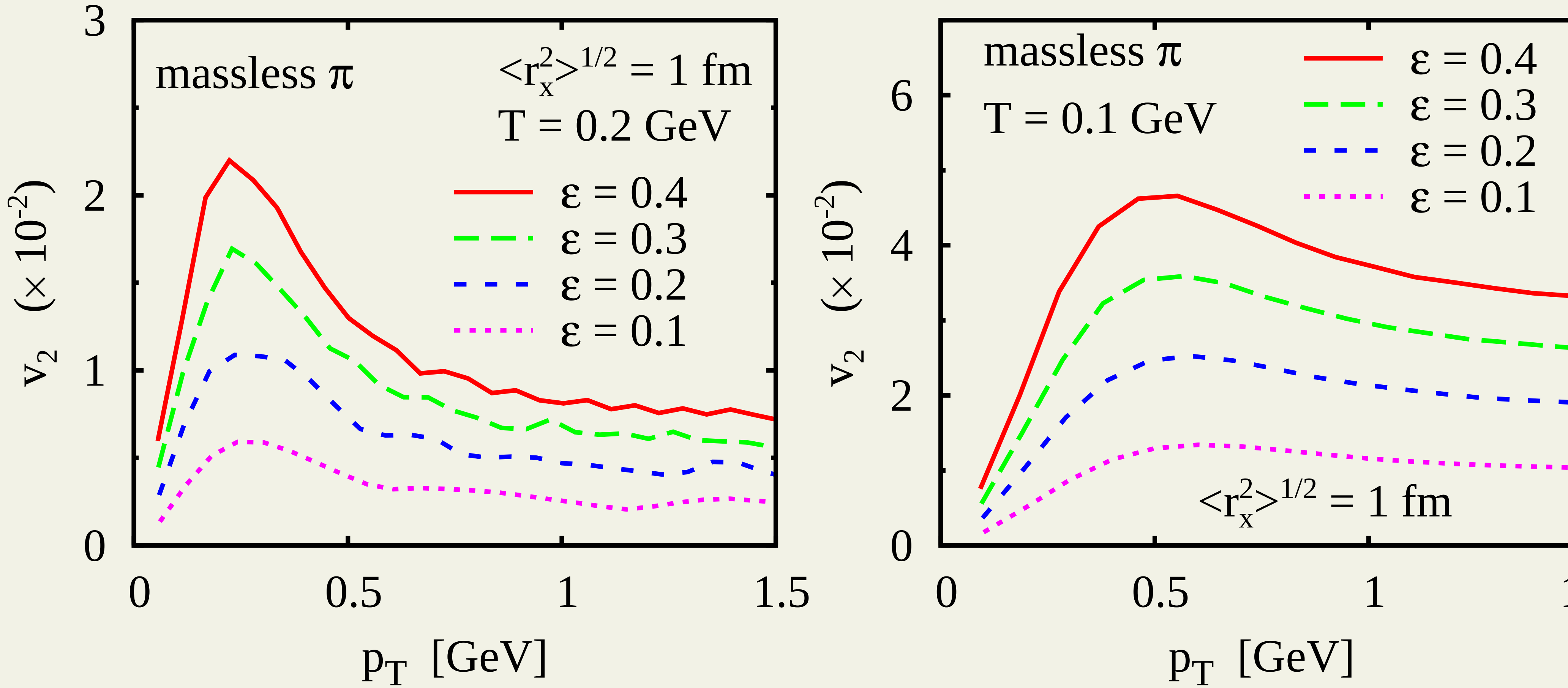


the intrinsic v_2 is smaller, and it is highest at lower momenta (linear $V(\bar{r})$ is less effiencient than quadratic $V(\bar{r})$ at keeping system small)

at high p_T, it saturates because tail of mom. distribution is no longer Gaussian (for Gaussian, $f(0, p_y{=}p_T)/f(p_x{=}p_T, 0) \sim e^{-p_T^2(a-b)} \to 0$, so $v_2(p_T) \to 1$)

Small systems and hydro freezeout

Still sizeable intrinsic anisotropy for small systems $R \sim 1$ fm and also at lower temperatures relevant for freezeout from hydrodynamics



Interactions

need insight into how the intrinsic anistropy combines with the anisotropy from subsequent dynamics (e.g., is it additive $v_2 = v_2^{intr} + v_2^{dyn}$, or nonlinear)

simple model: evolve thermal ensemble of states in a common, density dependent single-particle potential

$$V(\vec{r}, t) = K\rho(\vec{r}, t)$$

with $\rho(\vec{r}, t) = \frac{dN(\vec{r},t)}{d^3r}$ computed self-consistently in each time step

For small K one could, as first step, estimate V from the freely expanding $K = 0$ solution, which means solving the Schrödinger eqn with given $V(\vec{r}, t)$

We are pursuing this for the NR case because it may be possible to compute V analytically. Stay tuned...

Summary

Hydro is not the only source of momentum anisotropies. **Quantum systems with coordinate space anisotropy have, in general, momentum anisotropy** (Heisenberg uncertainty relation). The intrinsic anisotropy survives at finite temperature, and can play a role at both the initial conditions. and at freezeout. For nonrelativistic (NR) systems in a 2D harmonic oscillator potential, $v_2 \sim \epsilon / 12MT\langle r_x^2 \rangle$.

A recalculation for massless particles shows, in general, smaller anisotropies that depend similarly on system size ($v_2 \sim 1/L^2$), and saturate at modest $v_2 \sim \mathcal{O}(1\%)$ at high p_T instead of the rapid growth in the NR case. For small systems $R \sim 1$ fm and at low temperatures relevant to hydrodynamic freezeout, the anisotropy is still sizeable.

Because results between the massless and NR calculations change rather strongly, it is important to investigate next the intrinsic anisotropy for relativistic particles of nonzero mass.

Some open questions:
- proper generalization to local thermal equilibrium
- effect of the longitudinal expansion
- evolution of the anisotropy in an interacting system